

Solving a multi period portfolio problem with transaction costs

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Abstract

In this paper we show how one can find accurate analytical approximations to a dynamic investment strategy in the presence of transaction costs. In comparison with previous contributions, our approach has the significant advantage of allowing one to analytically calculate the quantiles of the final wealth distribution function. As the results of numerical example based on financial data show, our approximation procedure gives a fast and extremely accurate approximation to the solution. Extending on the result we show how the optimal rebalancing frequency and the optimal asset mix can be determined given some fixed amount of transaction cost charged for each transaction.

Keywords: dynamic portfolio selection, transaction costs, analytical approximation

1 Introduction

Since the now legendary paper by Markowitz (1952) there has been an explosion of papers on the subject of portfolio selection. Although the methodological advances had a tremendous effect on the academic community the impacts of the modelling approach were far less convincing within the financial industry. The main reasons for the lack of applicability can be attributed to the gap between theoretical model simplicity and the inherent real world complexity. Due to the above considerations, there remains significant room for the improvement of theoretical models on the topic of portfolio selection.

There are many ways to categorise investment strategies, one of the most common being the classification with respect to the amount of active trading each class of investment strategies imposes. Within static investment strategies an allocation between different asset classes is made only once; namely at the beginning of the investment period, whereas with dynamic strategies rebalancing or active trading is performed throughout the period.

One of the classes of dynamic investment strategies that has attracted enormous academic interest over the past decade is continuously rebalanced constant

mix strategies. Although very convenient from a theoretical viewpoint (Merton 1990), the applicability of constant mix investment strategies remains seriously in question. If the lack of exploration of favorable market dynamics is a serious drawback of static strategies then transaction costs must be the undoing of constant mix strategies. Since following market dynamics obviously comes at a price, one is undoubtedly confronted with the question of how to successfully incorporate both the negative impacts of transaction costs and explore the benefits of following the market dynamics by rebalancing portfolio allocations between different asset types.

In the financial literature there have been numerous attempts to account for the presence of transaction costs: Constantinides (1979) was one of the first to show that an investor facing transaction costs rebalances his portfolio less frequently than in a frictionless economy; Gennotte and Jung (1994) numerically solved the problem of an agent with utility defined preferences over terminal wealth and proportional transaction costs; Boyle et al. (1997) extended their approach by developing analytical expressions in the case the investor has a power utility function and the risky asset follows a multiplicative binomial process; Lowenstein (2000) examined the portfolio trading problem of an investor who faces transaction costs and short sales constraints in a continuous time economy; Rowland(1999) addressed the question of transaction costs and international portfolio diversification; while Marquering et al. (1999) examined the effect of transaction costs and habit persistence in explaining the cross-sectional variation in portfolio returns.

Although theoretical advances in terms of accounting for transaction costs have been made, it still proves extremely hard to find analytical solutions to realistic models with transaction costs.

One problem we confront when including transaction costs is that even under the simplest set of assumptions the models prove to be too complex to solve in an analytical form. Take as an example a simple periodically rebalanced two-asset model without any transaction costs, where one of the assets is risk-free and the other risky and follows a log-normal law. The end-period wealth is in this case a sum of log-normal dependent variables for which no analytical distribution function exists. The introduction of transaction costs inherently adds to the complexity of the problem.

In this Section I show how one can find accurate analytical approximations for a dynamic (constant mix)¹ investment strategy in the presence of transaction costs. Within my approach I account for transaction costs, which are assumed to be proportional to the amount of risky assets bought or sold. The assumption of proportional transaction costs is similar to most of the work done on the subject (Constantinides, (1986); Davis and Norman, (1990); Magill and Constantinides, (1976); Uppal, (1993).

In deriving the proposed approximation procedure I rely on the previous work by Dhaene et al. (2002_{a,b}), where the authors derive convex bounds for the distribution functions of otherwise inexpressible random variables or sums

¹Dynamic rebalancing does not automatically imply continuous rebalancing.

of random variables with no analytical description. In much the same way, I use the conditioning Taylor approach where a sum of log-normal dependent random variables is approximated by a conditional sum with the conditioning random variable set equal to the first-order Taylor expansion of the original sum.

Within this framework I explicitly work out a general example from the area of savings. I solve a problem of an investor who at time zero makes an investment that is being periodically rebalanced (on a yearly basis) according to a previously determined investment strategy (characterised by allocations between different asset classes) for a period of n years. In determining the optimal investment mix, I search for an asset mix that maximises the amount of $(1 - p)$ % guaranteed terminal wealth. Hence, my optimisation criteria is set equal to VaR (p%).

This approach of modelling an investor's preferences via VaR or more generally by a class of distortion functions is based on the concept introduced by Yaari (1987) and is in contrast to most of the financial literature on the subject; it is usual in finance to model an investor's preferences with utility functions. The main difference between the two approaches lies in the way they model an investor's preferences; in Yaari's theory, investors are assumed to take different positions on the probabilities of the outcome, (i.e. a conservative investor places more weight or a higher probability on the negative outcomes than the neutral probability distribution or physical measure suggests, whereas an aggressive investor overestimates the probabilities of positive or above-average outcomes; since in a way they both distort the neutral probability distribution this leads to the name distorted functions).

The organisation of this Section is as follows. In the following section, I introduce the basic properties of my model dynamics along with an investor's preferences and optimisation criteria. Section 5 explains the solution methodology. In Section 6 I explicitly work out a case of a single investment made at the beginning of the investment horizon. Numerical illustrations of the models considered are presented in Section 7, while in Section 8 I conclude.

2 Risk measures and comonotonicity

2.1 Risk measures

In this section I present some of the most important risk measures used in actuarial science. In simple terms, a risk measure is a functional that assigns a value to the distribution function of a random variable.

Perhaps the most important and widely used risk measure is Value at Risk or VaR. For a given random variable X the p -quantile risk measure (VaR) is defined by

$$Q_p[X] = \inf \{x \in \mathbb{R} \mid F_X(x) \geq p\}, \quad p \in (0, 1), \quad (1)$$

where $F_X(x) = \Pr[X \leq x]$. A related risk measure is denoted by $Q_p^+[X]$ and is

defined by

$$Q_p^+[X] = \sup \{x \in \mathbb{R} \mid F_X(x) \leq p\}, \quad p \in (0, 1). \quad (2)$$

Observe that only values of p corresponding to a horizontal segment of F_X lead to different values of $Q_p[X]$ and $Q_p^+[X]$. Thus when F_X is strictly increasing, both risk measures will coincide for all values of p . In this case, one can also define the $(1-p)$ -th quantiles by

$$Q_{1-p}[X] = \sup \{x \in \mathbb{R} \mid \bar{F}_X(x) \geq p\}, \quad p \in (0, 1), \quad (3)$$

where $\bar{F}_X(x) = 1 - F_X(x)$.

Another important risk measure often used is Tail Value-at-risk, denoted TVaR and defined as

$$TVaR_p = \frac{1}{1-p} \int_p^1 Q_q[X] \cdot dq; \quad p \in (0, 1), \quad (4)$$

Note that the TVaR can be interpreted as the expected value of quantiles from some p on.

Two risk measures closely related to TVaR are Conditional Tail Expectation and Expected Shortfall. The Conditional Tail Expectation denoted by (CTE) is defined as

$$CTE_p = E[X \mid X > Q_p[X]]; \quad p \in (0, 1). \quad (5)$$

The Expected Shortfall at level p denoted by (ESF_p) is defined by

$$ESF_p = E[(X - Q_p[X])_+]; \quad p \in (0, 1). \quad (6)$$

For more about the relationship between different risk measures see Dhaene et al. (2003).

2.2 Comonotonicity, comonotonic sets and comonotonic random vectors.

In this section I introduce the concepts of comonotonicity, comonotonic sets and comonotonic random vectors. First I state the definition of the comonotonic set (see e.g. Dhaene et al. (2003)).

Let \vec{x}, \vec{y} denote two random vectors in R^n and let $\vec{x} \leq \vec{y}$ denote componentwise order which is defined by $x_i \leq y_i$ for all $1 \leq i \leq n$.

Definition 1 *The set $A \subseteq R^n$ is said to be comonotonic if for any $\vec{x}, \vec{y} \in A$ the following relationship holds $\vec{x} \leq \vec{y}$ or $\vec{y} \leq \vec{x}$.*

Observe that from the definition it follows that any comonotonic set is simultaneously non-decreasing in each component. Thus a comonotonic set is a thin set, it cannot contain subsets with a dimension bigger than 1. Moreover, any subset of a comonotonic set is also comonotonic.

By having defined a comonotonic set I can proceed to define a comonotonic vector.

Definition 2 A random vector $\underline{Y} = (Y_1, Y_2, \dots, Y_n)$ is said to be comonotonic if

$$(Y_1, Y_2, \dots, Y_n) \stackrel{d}{=} (F_{Y_1}^{-1}(U), F_{Y_2}^{-1}(U), \dots, F_{Y_n}^{-1}(U)), \quad (7)$$

where U is a random variable which is uniformly distributed on the unit interval and where the notation $\stackrel{d}{=}$ stands for 'equality in distribution'.

For any random vector $\underline{X} = (X_1, X_2, \dots, X_n)$, I will call its comonotonic counterpart any random vector with the same marginal distributions and with the comonotonic dependency structure. The comonotonic counterpart of $\underline{X} = (X_1, X_2, \dots, X_n)$ will be denoted by $\underline{X}^c = (X_1^c, X_2^c, \dots, X_n^c)$. Hence for any random vector $\underline{X} = (X_1, X_2, \dots, X_n)$, one has

$$(X_1^c, X_2^c, \dots, X_n^c) \stackrel{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \dots, F_{X_n}^{-1}(U)). \quad (8)$$

It can be proven that a random vector is comonotonic if and only if all its marginals are non-decreasing functions (or all are non-increasing functions) of the same random variable.

3 Distortion functions

In this Section I explain the difference between utility functions and distortion functions as first introduced by Yaari in his dual theory of choice under risk (see Yaari (1987) or Wang & Young (1998)). In plain terms, utility functions represent an investor's preferences by assigning different weights to the outcomes of random variable (such as wealth), whereas in the distortion function context preferences are described by a distortion function which transforms the physical measure (or probability measure).

The distinction between both concepts can most effectively be presented by comparing the decision principles within both settings. More formally

Definition 3 A utility function U is a non-decreasing real-valued function on \mathbb{R} .

Within the utility theory the optimisation criterion is equal to the expected value of transformed random variable $U(X)$

$$\int_{-\infty}^{\infty} U(x) dF(x). \quad (9)$$

Definition 4 A distortion function g is a non-decreasing function $g : [0, 1] \rightarrow [0, 1]$ with $g(0) = 0$ and $g(1) = 1$.

In Yaari's dual theory of risk a decision principle is set equal to

$$- \int_{-\infty}^0 (1 - g(\bar{F}(x))) dx + \int_0^{\infty} g(\bar{F}(x)) dx, \quad (10)$$

with $\overline{F}(x)$ equal to

$$\overline{F}(x) = 1 - F(x). \quad (11)$$

Observe that although $g(\overline{F}(x))$ takes values on the interval $[0, 1]$ it cannot be regarded as a "new" probability measure (or distorted probability measure); namely $g(\overline{F}(x))$ will not necessarily be right continuous.

For the connection between risk measures and the distortion function, see Dhaene et al. (2003).

4 The model

I assume an investor with distortion function preferences defined over terminal wealth and a finite investment horizon of n years (for the purpose of numerical illustration $n=20$ is chosen). In considering investment strategies, I take the class of constant mix dynamically rebalanced strategies with inter-temporal rebalancing done on a periodic basis. I explicitly account for transaction costs with an additional term proportional to the amount of risky assets bought or sold. The optimality of an investment strategy is defined in terms of the maximisation of the VaR(p%) of the distribution function of terminal wealth. Although this approach of modelling an investor's preferences by means of VaR deviates from the standard procedure in the area of finance, this does not limit the applicability of my approach; my methodology can easily be extended to include utility defined preferences.

4.1 Portfolio dynamics of semi-static investment strategies under transaction costs

Consider a market of m risky and one risk-free security. The characteristics of the market are described by a joint probability density function f defined on the space $(0, \infty)^m \times \mathbb{Z}_+$

$$f(\vec{S}, k) = f(S_1(k), S_2(k), \dots, S_m(k)),$$

with \vec{S} denoting the values (prices) of one money unit investments in m risky securities in k -th period.² The distribution of log returns on individual securities is assumed to be unimodal and symmetrical and is assumed to belong to the class of stable distributions, while the log-return on well-diversified portfolios is assumed to be normally distributed (this will be the case if we consider returns of market index or a well-diversified benchmark over longer investment horizons).

²The returns from a single asset (logs of gross returns) are assumed to be i.i.d, so that the distribution of returns (or prices of securities) depends only on the length of the investment horizon.

Accordingly, a constant mix investment strategy is characterised by an allocation vector $\vec{\pi}$, between risky and risk-free assets, with π_i (i from 1 to m) denoting a percentage of wealth invested in risky asset i and $\pi_0(t) = 1 - \sum_{i=1}^m \pi_i(t)$ invested in the risk-free asset. In contrast to risky securities, the value of a money unit investment in the risk-free asset is deterministic and can be expressed as

$$S_{r,f}(t) = e^{\mu_{r,f} \cdot t},$$

where $\mu_{r,f}$ stands for the drift of the risk-free asset.

In the case of a periodically rebalanced constant mix strategy (with allocations done on a periodic basis), the end-period investment value is equal to

$$V_n = V_0 \prod_{j=1}^n \left(\pi_0 \cdot S_{r,f}(j) + \sum_{i=1}^m \pi_i \cdot S_i(j) \right), \quad (12)$$

where n is the number of times the investor readjusts his investment mix towards the required allocation strategy $\vec{\pi}$ and $S_i(j)$ represents the value of a one-unit investment in the risky security i in the j -th period.

In order to adequately capture all aspects of rebalancing, the equation must be modified to allow for the presence of transaction costs. I model the presence of transaction costs by an additional factor $T_i(\pi_i, C_i, S_i)$, which gives the net value of i -th security after the deduction of transaction costs

$$V_n = V_0 \prod_{j=1}^n \left(\pi_0 \cdot S_{r,f} + \sum_{i=1}^m T_i(\pi_i, C_i, S_i) \cdot \pi_i \cdot S_i(j) \right). \quad (13)$$

Here $T_i(\pi_i, C_i, S_i)$ is assumed to depend on the percentage of investment placed in the i -th risky asset π_i (for example, if π_i is equal to 1 there are no transaction costs since no rebalancing is required), the value of i -th risky asset S_i (transaction costs will depend on the end-period value of i -th risky asset and along with it on that period's realised return) and the amount of transaction costs charged for that security. Generalising the result (13) to the case of multi-period investment strategies with yearly investment amounts α_i and rebalancing done on a periodic basis, the terminal wealth is equal to

$$V_n = \sum_{k=1}^n \alpha_k \prod_{j=k}^n \left(\pi_0 \cdot S_{r,f} + \sum_{i=1}^m T_i(\pi_i, C_i, S_i) \cdot \pi_i \cdot S_i(j) \right). \quad (14)$$

A special case of interest is where there are only two assets available for investment, one risky and one risk-free. In this case, equation (14) simplifies to

$$V_n = \sum_{k=1}^n \alpha_k \prod_{j=k}^n (\pi_0 \cdot S_{r,f} + T_a(1 - \pi_0, C_a, S_a) \cdot (1 - \pi_0) \cdot S_a(j)), \quad (15)$$

where $S_{r,f}$ represents the value of the risk-free asset, and $S_a(j)$ the value of a one-unit investment in the risky security in the j -th period. Note that where

the risky investment is in a market index or a well-diversified portfolio then S_a can be assumed to be log-normally distributed.

4.2 Investor preferences and optimisation criterion

In modelling investor preferences I assume that they can be modelled within the class of distortion functions. The class of distortion functions was introduced by Yaari (1987) and can be considered as an alternative approach to the concept of utility theory. While in utility theory one assumes that investors assign utility levels to each outcome of the random variable (such as portfolio return), in Yaari’s dual theory of risk an investor’s preferences are described by means of distortion functions or “transformation” functions, which change the distribution of probability mass on the set of outcomes. In much the same way as in utility theory where one maximises the expected value of a utility transformed (assigned) random variable, under Yaari’s dual theory of risk optimality is achieved when the value of distorted distribution function is maximised

$$\max_{\bar{\pi}} \rho(V_n(\bar{\pi})).$$

Here ρ represents the risk measure associated with the distortion function and V_n the accumulated wealth (or terminal wealth) at the end of n periods. As mentioned at the beginning, I use Value-at-Risk (VaR) as the optimisation criterion. A similar choice of optimisation criterion is found in Dhaene et al. (2004_b).

It can be shown that maximising the VaR of the terminal wealth distribution function at a given probability level p yields an investment strategy that maximises the minimal amount of V_n achieved with a probability of at least $1 - p$. Thus, the optimisation criterion reads

$$\max_{\bar{\pi}} (F_{V_n(\bar{\pi})}^{-1}(p)), \tag{16}$$

with $F_{V_n(\bar{\pi})}^{-1}(p)$ denoting the inverse (or quantile) of the terminal wealth distribution function at probability p . Note that the problem of maximising a distortion risk measure or a quantile as the most distinct representative of this class of distortion risk measures is tightly linked to the problem of determining the distribution function of terminal wealth. One can see from equation (14) that even in a very simple model with only two assets and without transaction costs the terminal wealth is a sum of log-normally distributed random variables and hence the distribution function is impossible to determine analytically. Therefore, one has to rely on a Monte Carlo simulation which can be time-consuming, especially if one is interested in extreme quantiles of the distribution function. To make matters worse, when choosing among possible investment strategies due to the numerous securities available there is virtually an infinite number of possibilities to consider.

5 Solution methodology

5.1 Reducing the multidimensionality

One of the biggest obstacles to solving problems of the form (14) is the inherent multidimensionality that is encountered. As mentioned, due to the numerous number of assets available there is practically an infinite number of possible combinations one has to consider when analysing the problem. Therefore, it is very convenient if the multidimensionality of the problem can be reduced by proving that the solution to the problem can be found in a smaller subset than the original set. More specifically, I prove that the optimal investment strategy lies on the Capital Market Line (CML).

Theorem 5 *The solution to the problem (16) when only considering well-diversified portfolio strategies (with normal log-returns) is maximised by selecting an investment strategy from the CML.*

Before proving the stated result, I recall a helpful lemma.

Lemma 6 *Let \vec{Y} and \vec{Z} be two random vectors with non-negative independent components that are stochastically ordered (i.e. meaning that for each i $Y_i \geq_{st} Z_i$). Then for any non negative vector $\vec{\alpha}$, with non-negative components independent of \vec{Y} and \vec{Z} also $\vec{\alpha} \cdot \vec{Y}$ and $\vec{\alpha} \cdot \vec{Z}$ are stochastically ordered.*

Moreover, $\prod_{j=1}^n Y_j$ and $\prod_{j=1}^n Z_j$ are also stochastically ordered.

Proof. Define \vec{Y}^* , \vec{Z}^* to be random vectors of quantiles of marginals of Y_i , Z_i so that \vec{Y}^* can be written as a $\vec{Y}^* = (F_{Y_1}^{-1}(U_1), \dots, F_{Y_n}^{-1}(U_n))$ and $\vec{Z}^* = (F_{Z_1}^{-1}(U_1), \dots, F_{Z_n}^{-1}(U_n))$, where U_i are uniformly $(0, 1)$ distributed random variables. Obviously \vec{Y}^* has the same marginals as \vec{Y} and \vec{Z}^* has the same marginals as \vec{Z} . It is not hard to see that also

$$\vec{\alpha} \cdot \vec{Y} \stackrel{d}{=} \vec{\alpha} \cdot \vec{Y}^*$$

and

$$\prod_{j=1}^n Y_j \stackrel{d}{=} \prod_{j=1}^n Y_j^*,$$

while the same holds for r.v. \vec{Z} . Clearly

$$\vec{\alpha} \cdot \vec{Y}^* \geq_{st} \vec{\alpha} \cdot \vec{Z}^*,$$

since for any outcome (U_1, \dots, U_n) , $\vec{\alpha} \cdot \vec{Y}^*(U_1, \dots, U_n)$ is smaller than $\vec{\alpha} \cdot \vec{Z}^*(U_1, \dots, U_n)$. Obviously this suffices since the U_i 's a.s. determine the set of all outcomes. In a similar way, one can also prove that

$$\prod_{j=1}^n Y_j \geq_{st} \prod_{j=1}^n Z_j.$$

■

I can now proceed to proving the stated result of the theorem

Proof. First note that it suffices to prove the stochastic dominance of CML investment strategies. Namely for any type of optimisation criteria the following relationship holds

$$X \leq_{st} Y \Rightarrow \eta[X] \leq \eta[Y], \quad (17)$$

where η denotes the optimisation criterion (e.g. expected utility function) defined over non-negative random variables X, Y (e.g. terminal wealth). In simple terms if one can prove that the distribution function of end wealth under CML investment strategy stochastically dominates all other distribution functions arising from other investment strategies (ones not on the CML) this allows one to restrict himself to CML portfolios.

Recall that CML is defined as a subset F of the half-plane $R_+^2 = \{(\mu, \sigma) | \mu \in R, \sigma \geq 0\}$ which corresponds to all mean-variance efficient portfolios. More precisely, all CML portfolios have the following property

$$\min_{\vec{\pi}} \sigma(\vec{\pi}) \quad \text{subject to} \quad \mu(\vec{\pi}) = \mu, \quad (18)$$

which can in an alternative form be rewritten as

$$\max_{\vec{\pi}} \mu(\vec{\pi}) \quad \text{subject to} \quad \sigma(\vec{\pi}) = \sigma. \quad (19)$$

Observe that each of the yearly log-returns Y_i can be written as

$$Y_i(\vec{\pi}) = \mu(\vec{\pi}) + \sigma(\vec{\pi}) \cdot \Phi^{-1}(U_i),$$

where U_i are independent uniform random variables and Φ^{-1} is the inverse of standardised normal c.d.f. Thus, the yearly return from a CML investment strategy stochastically dominates the yearly returns of other investment strategies

$$Y_i(\vec{\pi}) \underset{\vec{\pi} \in CML}{\geq_{st}} Y_i(\vec{\pi}).$$

Since end-period wealth is simply just a sum of the products of exponents of yearly returns by lemma 6 the stochastic dominance of the end-wealth of investment strategies from the CML follows. ■

5.2 Approximating procedure

Consider a sum of log-normal variables

$$S = \sum_{i=1}^n \alpha_i \cdot e^{Z(i)}, \quad (20)$$

with α_i positive constants and $\vec{Z} = (Z(1), Z(2), \dots, Z(n))$ a random vector with drift $\vec{\mu}$ and variance covariance matrix Υ . One can conveniently assume that α_i

represents the amount invested i -th year and $Z(i)$ the investment return over horizon of $n - i$ years long (from year i to year n). Thus S can be taken to represent the accumulated value (or future value) of a series of periodic investments.

In general (if at least two α_i are non-zero) the distribution function of S cannot be determined analytically since (20) is a sum of dependent lognormal variables. One way of approaching this problem is to use a Taylor series expansion up to the first order which allows the distribution function of S to be written in an analytically tractable form. Although convenient, a first-order Taylor approximation will only work well if the variances of $Z(i)$ are relatively small, which will rarely be the case. Due to the severe limitations of a first-order Taylor approximation, one seeks for a method that is more robust and more widely applicable

In confronting this problem, Dhaene et al. (2002_{a,b}) have shown that by using the technique of conditioning on the first-order Taylor approximation one gets an approximating sequence that is at the same time analytically tractable as well as highly accurate. Following their methodology, it can be shown that any sum of the form (20) can be approximated by a sequence:

$$E[S | \Lambda] = \sum_{i=1}^n \alpha_i \cdot E[e^{Z(i)} | \Lambda], \quad (21)$$

with E denoting the expectation operator, and the conditioning variable³ Λ defined as

$$\Lambda = \sum_{i=1}^n \alpha_i \cdot e^{\mu_i} \cdot Z(i), \quad (22)$$

with μ_i denoting the drift rate or the expected value of $Z(i)$. Observe that expression (22) is in fact a first-order Taylor series expression of the original sum around expected values of $Z(i)$ (i.e. if one derives a first-order approximation of (21) around the expected values of $Z(i)$; the resulting expression is up to a constant equal to Λ) so that the conditioning is done with respect to the approximated value of S derived by the first-order Taylor series expansion.

In comparing both methods (Taylor and conditioning), one may ask why is it more suitable and accurate to use the latter. Clearly the first-order Taylor approximation suffers from a severe problem of misspecification of the terms appearing in S ; where a sum of log-normally distributed random variables is replaced by the sum of normally distributed variables (i.e. each log-normal term is replaced by its normally distributed counterpart). In comparison, the conditioning technique gives both an accurate description of the stochastic process by accurately capturing the statistics of the return vector with the choice of the conditioning random variable Λ (i.e. note that the set of events or possible realisations of the return vector perfectly determines the set of all outcomes of S) and at the same avoids the problem of misspecification of the terms involved in

³For more on choosing the appropriate conditioning random variable, see Vanduffel et al. (2004) and Ahčan (2005_{a,b})

S , with each term in the sum being replaced (approximated) by its counterpart that is again log-normally distributed. As will soon be shown, besides being significantly more accurate the conditioning approximating procedure provides an analytic formulation of the quantiles of the approximating distribution function.

By taking into account the functional form of Λ and considering that the characteristics of the returns $Z(i)$ depend on the investment strategy $\boldsymbol{\pi}$, (21) can be written out as a:

$$E[S | \Lambda] = \sum_{i=0}^n \alpha_i e^{(n-i) \mu(\boldsymbol{\pi}) + \frac{1}{2}(1-r_i^2(\boldsymbol{\pi})) (n-i) \sigma^2(\boldsymbol{\pi}) + r_i(\boldsymbol{\pi}) \sqrt{n-i} \sigma(\boldsymbol{\pi}) \Phi^{-1}(U)}, \quad (23)$$

with U a standard Uniform random variable on $(0, 1)$ and $r_i(\boldsymbol{\pi})$ given by

$$r_i(\boldsymbol{\pi}) = \frac{\sum_{j=i+1}^n \sum_{k=0}^{j-1} \alpha_k e^{k \mu(\boldsymbol{\pi})}}{\sqrt{n-i} \sqrt{\sum_{j=1}^n \left(\sum_{k=0}^{j-1} \alpha_k e^{k \mu(\boldsymbol{\pi})} \right)^2}}. \quad (24)$$

Note that the correlation coefficients $r_i(\boldsymbol{\pi})$ are non-negative, implying that the sum in (23) is strictly increasing in $\Phi^{-1}(U)$. It thus follows directly that the quantiles of the distribution function of the approximating sequence are given by

$$Q_{1-p}(E[S | \Lambda]) = \sum_{i=0}^n \alpha_i e^{(n-i) \mu(\boldsymbol{\pi}) + \frac{1}{2}(1-r_i^2(\boldsymbol{\pi})) (n-i) \sigma^2(\boldsymbol{\pi}) - r_i(\boldsymbol{\pi}) \sqrt{n-i} \sigma(\boldsymbol{\pi}) \Phi^{-1}(p)}, \quad (25)$$

with Q_{1-p} denoting the quantile of the distribution function of $E[S | \Lambda]$.

6 Applications

6.1 The case of a single investment

Consider a simple case where a single investment of 1 unit is made at the beginning of the investment period (i.e at time 0). For the purpose of the numerical evaluation I take the length of the investment horizon to be equal to 20 years, whereby rebalancing is done periodically at the beginning of each year. As in subsection 4.1, I introduce transaction costs that are, as before, taken into account by the additional factor $T_i(\pi_i, C_i, S_i)$. Under this set of assumptions the distribution function of terminal wealth is equal to (13), with V_0 set equal to 1

$$V_n = \prod_{j=1}^n \left(\pi_0 \cdot S_{rf} + \sum_{i=1}^m T_i(\pi_i, C_i, S_i) \cdot \pi_i \cdot S_i(j) \right). \quad (26)$$

As the optimisation criteria (VaR) belongs to the class of distortion risk measures, I only need to consider investment strategies from the CML (i.e. the allocations are made between the risk-free account and the market portfolio;

where I assume the price of the market portfolio is log-normally distributed $S_a(j) = e^{Y_a(j)}$. Thus, the end-period portfolio value can be given as

$$V_n = \prod_{j=1}^n (\pi_0 \cdot S_{rf} + (1 - \pi_0) \cdot S_a(j) - \pi_0 \cdot (1 - \pi_0) \cdot C \cdot |S_a(j) - S_{rf}|), \quad (27)$$

with $S_a(j)$ value (gross return) of a one money unit investment in the market portfolio in the j -th period and the last term $\pi_0 \cdot (1 - \pi_0) \cdot C \cdot |S_a(j) - S_{rf}|$ representing the cost of rebalancing. Here C denotes transaction costs expressed as a percentage of the total amount rebalanced, whereas $\pi_0 \cdot (1 - \pi_0) \cdot |S_a(j) - S_{rf}|$ gives the amount that has to be either bought or sold in order to maintain the desired asset mix. Note that the inclusion of transaction costs inherently changes the statistical properties of the end-period portfolio value equation. Namely if transaction costs are excluded then (27) can be written in the form of a sum:

$$V_n^* = \sum_{i=1}^n ((\pi_0 \cdot S_{rf})^i \cdot ((1 - \pi_0))^{n-i} \cdot P_{S_a}(n-i)), \quad (28)$$

where the last term $P_{S_a}(n-i)$ gives the sum of all possible combinations obtained by multiplying $(n-i)$ terms from the set $(S_a(1), \dots, S_a(n))$. Observe that the number of terms of $P_{S_a}(n-i)$ is equal to $\frac{n!}{(n-i)! \cdot i!}$ and may be very large, especially if the rebalancing is done quite frequently within the investment horizon (i.e if n is large). $P_{S_a}(n-i)$ takes the simplest form when $(n-i)$ is equal to 1 or n .

$$P_{S_a}(1) = S_a(1) + S_a(2) + \dots + S_a(n), \quad (29)$$

$$P_{S_a}(n) = S_a(1) \cdot S_a(2) \cdot \dots \cdot S_a(n). \quad (30)$$

From equation (28) one can see that the reduced form of terminal wealth is equal to the sum of dependent log-normally distributed random variables. Hence, the technique of conditioning Taylor sum could apply if it were not for transaction costs, which introduce intractable nonlinearity into our model. Luckily, there is a way to overcome these problems. First observe that transaction costs, since $C \ll 1$ and $\pi_0 \cdot (1 - \pi_0) \leq 0.25$, represent only a small disturbance to the realised one-period gross return and consequently to the reduced form distribution function. Even more importantly, transaction costs are proportional to the absolute difference between the risky and risk-free value and therefore the quantiles of the reduced form distribution function are not “significantly” perturbed by the introduction of transaction costs. Accordingly, one can assume that the quantiles of the terminal wealth distribution function (with transaction costs) can be obtained by a simple modification (that takes into account the transaction costs effect) of the reduced form quantiles. The procedure of calculating the p -quantile will therefore consist of the calculation of the p -quantile of reduced form distribution function which will be “corrected” for the presence of transaction costs.

In line with this approach, one first has to find an approximating sequence for the reduced form equation (28). As mentioned in Subsection 5.2, the quality of the approximating sequence depends heavily on the choice of conditioning random variable. Here the conditioning random variable should be chosen that best reflects the statistical properties of the sum in (28).

There are two ways to proceed to obtaining the most suitable conditioning variable. Under the general approach one can use a Taylor-series expansion up to the first order and obtain an approximating procedure for the sum in (28). Alternatively, an appropriate conditioning variable can be obtained by examining the mathematical properties of the sum of interest (28). Since expressions (27), (28) are symmetrical with regard to yearly market prices $S_a(j)$ (a simple perturbation of any of the two indexes does not change the properties of the d.f.) the conditioning random variable Λ should also be symmetrical in yearly returns $Y_a(i)$ to best reflect the properties of the original sum

$$\Lambda = Y_a(1) + Y_a(2) + \dots + Y_a(n). \quad (31)$$

With this choice of random variable the approximating sequence is equal to

$$V_n^*(U) = \sum_{i=1}^n \frac{n!}{(n-i)! \cdot i!} \cdot (\pi_0 \cdot S_{rf})^i \cdot (1 - \pi_0)^{n-i} \cdot e^{(n-i) \mu(\boldsymbol{\pi}) - \frac{1}{2} r_i^2(\boldsymbol{\pi}) (n-i) \sigma^2(\boldsymbol{\pi}) + r_i(\boldsymbol{\pi}) \sqrt{n-i} \sigma(\boldsymbol{\pi})} \Phi^{-1}(U), \quad (32)$$

with U as before $(0,1)$ uniform random variable and r_i equal to

$$r_i(\boldsymbol{\pi}) = \frac{i}{\sqrt{n \cdot i}}. \quad (33)$$

In my second step, I need to correct the reduced form approximating procedure to account for the presence of transaction costs. After some calculations one can show that it makes sense to use the following approximation procedure

$$V_n(U) = V_n^*(U) - n \cdot \pi_0 \cdot (1 - \pi_0) \cdot C \cdot \left| e^{\mu + \sigma \Phi^{-1}(U)} - S_{rf} \right| \cdot V_{n-1}^*(U). \quad (34)$$

Here $V_n^*(U)$ represents the end wealth without transaction costs (32), while $\pi_0 \cdot (1 - \pi_0) \cdot C \cdot \left| e^{\mu + \sigma \Phi^{-1}(U)} - S_{rf} \right| \cdot V_{n-1}^*(U)$ gives a first-order approximation of the transaction costs involved. It is straightforward that the quantiles of $V_n^*(U)$ can be expressed as

$$Q_p(V_n(U)) = Q_p(V_n^*(U)) - n \cdot \pi_0 \cdot (1 - \pi_0) \cdot C \cdot \left| e^{\mu + \sigma \Phi^{-1}(p)} - S_{rf} \right| \cdot Q_p(V_{n-1}^*(U)). \quad (35)$$

7 Numerical illustration

In this section I present the results of numerical calculations for the model presented in the previous section. For the purpose of simulation, I fix the

number of years to $N = 20$. In explicitly solving the model, I choose the following set of parameters; drift of the market index $\mu_m = 0.07$ and standard deviation $\sigma_m = 0.16$, drift of the risk-free asset $\mu_r = 0.014$ and transaction costs $C = 0.01$. The choice of parameters corresponds to the majority of financial literature on the subject (Campbell (2002), Brennan (1997)).

In Figure 1 I present the results of the numerical simulation where the optimisation rule was set equal to the 0.05-quantile (VaR at 5% probability). As one can see from the figure, my approximating procedure gives an excellent fit against the Monte Carlo simulation. The maximum deviation between the approximating procedure (circles) and the Monte Carlo simulation (solid line) is negligible (0.5%).

The optimal investment strategy (characterised by an allocation between risky and risk-free assets, where the second value represents the fraction of wealth invested in risky assets) obtained by means of an approximating procedure is equal to $\pi_{app} = (52\%, 48\%)$, compared to the optimal asset mix obtained by means of a Monte Carlo $\pi_{MC} = (52.5\%, 47.5\%)$. The corresponding end-period wealths are equal to 13.28 money units in the case of the approximating sequence and 13.24 in the case of the Monte Carlo simulation.

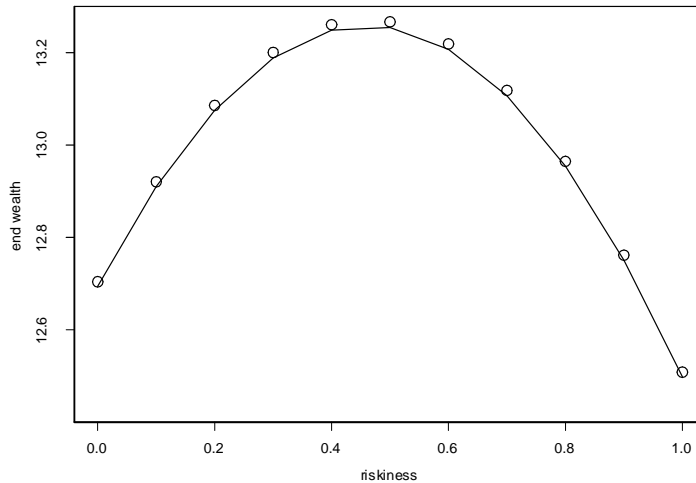


Figure 1: The approximated 0.05-quantile of terminal wealth versus the simulated 0.05-quantile of terminal wealth as a function of the risky proportion (with transaction cost).

In analysing the results one can observe that due to the quality of fit optimality is in both cases achieved for the same investment strategy, with the mismatch between actual value and the value obtained by the approximating

procedure being highest for those strategies with an equal division between risky and risk-free assets. This is not at all surprising since the expected transaction costs are highest in case of such strategies.

8 Final remarks

In this Section I explicitly solved a dynamic portfolio selection problem with transaction costs. In comparison with previous contributions, my approach has a significant advantage of allowing one to analytically calculate the quantiles of the end-period wealth distribution function. As the results of numerical example based on real life data show, my approximating procedure gives a fast and extremely accurate approximation of the analysed problem.

Even though my procedure offers a significant step forward in the analysis of multiperiod portfolio problems within the classical framework, several problems remain to be answered in the future. The first and most obvious problem currently being worked on is how to generalise these results to the case of periodic investment strategies with yearly investments in the capital market; money is invested throughout the investment period in reverse to a single initial investment. In this case, the distribution function of end-period wealth is no longer symmetrical with regard to the yearly returns and the approximating sequence does not have as suitable properties as in the case of a single initial investment. Another problem is linked to the choice of the distribution function describing the returns; if rebalancing is done on a more frequent basis than one year, the choice of lognormally distributed asset prices will prove too limiting and one needs to describe asset returns in terms of Levy processes. A less serious limitation of my work is that I do not use utility defined preferences. These can be easily included in my model, since the approximating sequence adequately describes the distribution function of end-period wealth thereby also providing accurate approximations in the case of utility defined preferences.

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